

# MARGINALLY TRAPPED SURFACES WITH POINTWISE 1-TYPE GAUSS MAP IN MINKOWSKI 4-SPACE

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**ABSTRACT.** A marginally trapped surface in the four-dimensional Minkowski space is a spacelike surface whose mean curvature vector is lightlike at each point. In the present paper we find all marginally trapped surfaces with pointwise 1-type Gauss map. We prove that a marginally trapped surface is of pointwise 1-type Gauss map if and only if it has parallel mean curvature vector field.

## 1. INTRODUCTION

The concept of trapped surfaces was introduced by Roger Penrose [22] and it plays an important role in General Relativity for studying singularities and also for understanding the evolution of black holes, the cosmic censorship hypothesis, the Penrose inequality, etc. In Physics, a surface in a 4-dimensional spacetime is called marginally trapped if it is closed, embedded, spacelike and its mean curvature vector is lightlike at each point of the surface. Recently, marginally trapped surfaces have been studied from a mathematical viewpoint. In the mathematical literature, it is customary to call a surface *marginally trapped* or *quasi-minimal* [23, 5] if its mean curvature vector  $H$  is lightlike at each point, and removing the other hypotheses, i.e. the surface does not need to be closed or embedded.

Classification results in 4-dimensional Lorentz space forms were obtained imposing some extra conditions on the mean curvature vector, the Gauss curvature or the second fundamental form. For example, marginally trapped surfaces with positive relative nullity in Lorentz space forms were classified by B.-Y. Chen and J. Van der Veken [7]. They also proved the non-existence of marginally trapped surfaces in Robertson-Walker spaces with positive relative nullity [8] and classified marginally trapped surfaces with parallel mean curvature vector in Lorentz space forms [9].

In [16] S. Haesen and M. Ortega classified marginally trapped surfaces in Minkowski 4-space which are invariant under spacelike rotations. Marginally trapped surfaces in Minkowski 4-space which are invariant under boost transformations (hyperbolic rotations) were classified in [15] and marginally trapped surfaces which are invariant under the group of screw rotations (a group of Lorentz rotations with an invariant lightlike direction) were studied in [17].

In [14] G. Ganchev and the present author developed the invariant theory of marginally trapped surfaces in the four-dimensional Minkowski space  $\mathbb{R}_1^4$ . Our approach to the study of these surfaces was based on the principal lines generated by the second fundamental form. Using the principal lines, we introduced a geometrically determined moving frame field at each point of such a surface. The derivative formulas for this frame field imply the

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existence of seven invariant functions. We proved that each marginally trapped surface is determined up to a motion in  $\mathbb{R}_1^4$  by these seven invariant functions satisfying some natural conditions.

In the present paper we express the Laplacian of the Gauss map of a marginally trapped surface in terms of these invariant functions. Imposing the condition that the surface has pointwise 1-type Gauss map, we obtain that three of the invariants are zero. We give necessary and sufficient conditions for a marginally trapped surface to have pointwise 1-type Gauss map and find all marginally trapped surfaces with pointwise 1-type Gauss map. Our main result states that a marginally trapped surface is of pointwise 1-type Gauss map if and only if it has parallel mean curvature vector field.

## 2. INVARIANTS OF A MARGINALLY TRAPPED SURFACE

Let  $\mathbb{R}_1^4$  be the Minkowski space endowed with the metric  $\langle \cdot, \cdot \rangle$  of signature  $(3, 1)$  and  $Oe_1e_2e_3e_4$  be a fixed orthonormal coordinate system in  $\mathbb{R}_1^4$  such that  $e_1^2 = e_2^2 = e_3^2 = 1$ ,  $e_4^2 = -1$ . The standard flat metric is given in local coordinates by  $dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2$ .

A surface  $M^2$  in  $\mathbb{R}_1^4$  is said to be *spacelike* if  $\langle \cdot, \cdot \rangle$  induces a Riemannian metric  $g$  on  $M^2$ , i.e. at each point  $p$  of a spacelike surface  $M^2$  we have the following decomposition

$$\mathbb{R}_1^4 = T_p M^2 \oplus N_p M^2$$

with the property that the restriction of the metric  $\langle \cdot, \cdot \rangle$  onto the tangent space  $T_p M^2$  is of signature  $(2, 0)$ , and the restriction of the metric  $\langle \cdot, \cdot \rangle$  onto the normal space  $N_p M^2$  is of signature  $(1, 1)$ .

We denote by  $\nabla'$  and  $\nabla$  the Levi Civita connections on  $\mathbb{R}_1^4$  and  $M^2$ , respectively. Let  $x$  and  $y$  be vector fields tangent to  $M$  and let  $\xi$  be a normal vector field. Then the formulas of Gauss and Weingarten give decompositions of the vector fields  $\nabla'_x y$  and  $\nabla'_x \xi$  into tangent and normal components:

$$\begin{aligned}\nabla'_x y &= \nabla_x y + \sigma(x, y); \\ \nabla'_x \xi &= -A_\xi x + D_x \xi,\end{aligned}$$

which define the second fundamental tensor  $\sigma$ , the normal connection  $D$  and the shape operator  $A_\xi$  with respect to  $\xi$ . The mean curvature vector field  $H$  of  $M^2$  is defined as  $H = \frac{1}{2} \text{tr } \sigma$ . Thus, if  $M^2$  is a spacelike surface and  $\{x, y\}$  is a local orthonormal frame of the tangent bundle, the mean curvature vector field is given by the formula  $H = \frac{1}{2} (\sigma(x, x) + \sigma(y, y))$ .

Let  $M^2 : z = z(u, v)$ ,  $(u, v) \in \mathcal{D}$  ( $\mathcal{D} \subset \mathbb{R}^2$ ) be a local parametrization on a spacelike surface in  $\mathbb{R}_1^4$ . The tangent space at an arbitrary point  $p$  of  $M^2$  is  $T_p M^2 = \text{span}\{z_u, z_v\}$ . Since  $M^2$  is spacelike,  $\langle z_u, z_u \rangle > 0$ ,  $\langle z_v, z_v \rangle > 0$ . We use the standard denotations  $E(u, v) = \langle z_u, z_u \rangle$ ,  $F(u, v) = \langle z_u, z_v \rangle$ ,  $G(u, v) = \langle z_v, z_v \rangle$  for the coefficients of the first fundamental form

$$I(\lambda, \mu) = E\lambda^2 + 2F\lambda\mu + G\mu^2, \quad \lambda, \mu \in \mathbb{R}$$

and set  $W = \sqrt{EG - F^2}$ . Let us choose a normal frame field  $\{n_1, n_2\}$  such that  $\langle n_1, n_1 \rangle = 1$ ,  $\langle n_2, n_2 \rangle = -1$ , and the quadruple  $\{z_u, z_v, n_1, n_2\}$  is positively oriented in  $\mathbb{R}_1^4$ . Then we have

the following derivative formulas:

$$\begin{aligned}\nabla'_{z_u} z_u &= z_{uu} = \Gamma_{11}^1 z_u + \Gamma_{11}^2 z_v + c_{11}^1 n_1 - c_{11}^2 n_2; \\ \nabla'_{z_u} z_v &= z_{uv} = \Gamma_{12}^1 z_u + \Gamma_{12}^2 z_v + c_{12}^1 n_1 - c_{12}^2 n_2; \\ \nabla'_{z_v} z_v &= z_{vv} = \Gamma_{22}^1 z_u + \Gamma_{22}^2 z_v + c_{22}^1 n_1 - c_{22}^2 n_2,\end{aligned}$$

where  $\Gamma_{ij}^k$  are the Christoffel's symbols and the functions  $c_{ij}^k$ ,  $i, j, k = 1, 2$  are given by

$$\begin{aligned}c_{11}^1 &= \langle z_{uu}, n_1 \rangle; & c_{12}^1 &= \langle z_{uv}, n_1 \rangle; & c_{22}^1 &= \langle z_{vv}, n_1 \rangle; \\ c_{11}^2 &= \langle z_{uu}, n_2 \rangle; & c_{12}^2 &= \langle z_{uv}, n_2 \rangle; & c_{22}^2 &= \langle z_{vv}, n_2 \rangle.\end{aligned}$$

Obviously, the surface  $M^2$  lies in a 2-plane if and only if  $M^2$  is totally geodesic, i.e.  $c_{ij}^k = 0$ ,  $i, j, k = 1, 2$ . So, we assume that at least one of the coefficients  $c_{ij}^k$  is not zero.

Let  $X = \lambda z_u + \mu z_v$ ,  $(\lambda, \mu) \neq (0, 0)$  be a tangent vector at a point  $p \in M^2$ . The second fundamental form  $II$  of the surface  $M^2$  at the point  $p$  is introduced by the formula

$$II(\lambda, \mu) = L\lambda^2 + 2M\lambda\mu + N\mu^2,$$

where the functions  $L$ ,  $M$ , and  $N$  are defined as follows:

$$L = \frac{2}{W} \begin{vmatrix} c_{11}^1 & c_{12}^1 \\ c_{11}^2 & c_{12}^2 \end{vmatrix}; \quad M = \frac{1}{W} \begin{vmatrix} c_{11}^1 & c_{22}^1 \\ c_{11}^2 & c_{22}^2 \end{vmatrix}; \quad N = \frac{2}{W} \begin{vmatrix} c_{12}^1 & c_{22}^1 \\ c_{12}^2 & c_{22}^2 \end{vmatrix}.$$

The second fundamental form  $II$  is invariant up to the orientation of the tangent space or the normal space of the surface.

The condition  $L = M = N = 0$  characterizes points at which the space  $\{\sigma(x, y) : x, y \in T_p M^2\}$  is one-dimensional. We call such points *flat points* of the surface [12, 13]. These points are analogous to flat points in the theory of surfaces in  $\mathbb{R}^3$ . In [18] and [19] such points are called inflection points. The notion of an inflection point is introduced for 2-dimensional surfaces in a 4-dimensional affine space  $\mathbb{A}^4$ . E. Lane [18] has shown that every point of a surface in  $\mathbb{A}^4$  is an inflection point if and only if the surface is developable or lies in a 3-dimensional space. Further we consider surfaces free of flat points, i.e.  $(L, M, N) \neq (0, 0, 0)$ .

The second fundamental form  $II$  determines conjugate, asymptotic, and principal tangents at a point  $p$  of  $M^2$  in the standard way. A line  $c : u = u(q)$ ,  $v = v(q)$ ;  $q \in J \subset \mathbb{R}$  on  $M^2$  is said to be a *principal line*, if its tangent at any point is principal.

It is interesting to note that the "umbilical" points, i.e. points at which the coefficients of the first and the second fundamental forms are proportional ( $L = \rho E$ ,  $M = \rho F$ ,  $N = \rho G$ ,  $\rho \neq 0$ ), are exactly the points at which the mean curvature vector  $H$  is zero. So, the spacelike surfaces consisting of "umbilical" points in  $\mathbb{R}_1^4$  are exactly the surfaces with zero mean curvature. If  $M^2$  is a spacelike surface free of "umbilical" points ( $H \neq 0$  at each point), then there exist exactly two principal tangents.

Now, let  $M^2 : z = z(u, v)$ ,  $(u, v) \in \mathcal{D}$  be a marginally trapped surface. Then the mean curvature vector is lightlike at each point of the surface, i.e.  $\langle H, H \rangle = 0$ ,  $H \neq 0$ . Hence there exists a pseudo-orthonormal normal frame field  $\{n_1, n_2\}$ , such that  $n_1 = H$  and

$$\langle n_1, n_1 \rangle = 0; \quad \langle n_2, n_2 \rangle = 0; \quad \langle n_1, n_2 \rangle = -1.$$

We assume that  $M^2$  is free of flat points, i.e.  $(L, M, N) \neq (0, 0, 0)$ . Then at each point of the surface there exist principal lines and without loss of generality we assume that  $M^2$  is parameterized by principal lines. Let us denote  $x = \frac{z_u}{\sqrt{E}}$ ,  $y = \frac{z_v}{\sqrt{G}}$ . Thus we obtain a special frame field  $\{x, y, n_1, n_2\}$  at each point  $p \in M^2$ , such that  $x, y$  are unit spacelike vector fields collinear with the principal directions;  $n_1, n_2$  are lightlike vector fields,  $\langle n_1, n_2 \rangle = -1$ , and  $n_1$  is the mean curvature vector field. We call this frame field a *geometric frame field* of  $M^2$ .

With respect to the geometric frame field we have the following derivative formulas of  $M^2$ :

$$(1) \quad \begin{aligned} \nabla'_x x &= \gamma_1 y + (1 + \nu) n_1; & \nabla'_x n_1 &= \mu y + \beta_1 n_1; \\ \nabla'_x y &= -\gamma_1 x + \lambda n_1 + \mu n_2; & \nabla'_y n_1 &= \mu x + \beta_2 n_1; \\ \nabla'_y x &= -\gamma_2 y + \lambda n_1 + \mu n_2; & \nabla'_x n_2 &= (1 + \nu) x + \lambda y - \beta_1 n_2; \\ \nabla'_y y &= \gamma_2 x + (1 - \nu) n_1; & \nabla'_y n_2 &= \lambda x + (1 - \nu) y - \beta_2 n_2, \end{aligned}$$

where  $\nu = -\langle \frac{\sigma(x, x) - \sigma(y, y)}{2}, n_2 \rangle$ ,  $\lambda = -\langle \sigma(x, y), n_2 \rangle$ ,  $\mu = -\langle \sigma(x, y), n_1 \rangle$ ,  $\gamma_1 = -y(\ln \sqrt{E})$ ,  $\gamma_2 = -x(\ln \sqrt{G})$ ,  $\beta_1 = -\langle \nabla'_x n_1, n_2 \rangle$ ,  $\beta_2 = -\langle \nabla'_y n_1, n_2 \rangle$ .

Note that the functions  $\nu, \lambda, \mu, \gamma_1, \gamma_2, \beta_1, \beta_2$  are invariants of the surface determined by the principal directions. In [14] we proved the fundamental theorem for marginally trapped surfaces in  $\mathbb{R}_1^4$ , which states that each marginally trapped surface free of flat points is determined up to a motion in  $\mathbb{R}_1^4$  by these seven invariant functions satisfying some natural conditions.

Formulas (1) imply that the Gauss curvature  $K$  and normal curvature  $\varkappa$  of  $M^2$  are expressed by the functions  $\nu, \lambda$ , and  $\mu$  as follows:

$$(2) \quad K = 2\lambda\mu; \quad \varkappa = -2\mu\nu.$$

### 3. SURFACES WITH POINTWISE 1-TYPE GAUSS MAP

An isometric immersion  $x : M \rightarrow \mathbb{E}^m$  of a submanifold  $M$  in the Euclidean space  $\mathbb{E}^m$  or pseudo-Euclidean space  $\mathbb{E}_s^m$  with signature  $(s, m - s)$  is said to be of *finite type* [3], if  $x$  identified with the position vector field of  $M$  in  $\mathbb{E}^m$  or  $\mathbb{E}_s^m$  can be expressed as a finite sum of eigenvectors of the Laplacian  $\Delta$  of  $M$ , i.e.

$$x = x_0 + \sum_{i=1}^k x_i,$$

where  $x_0$  is a constant map,  $x_1, x_2, \dots, x_k$  are non-constant maps such that  $\Delta x_i = \lambda_i x_i$ ,  $\lambda_i \in \mathbb{R}$ ,  $1 \leq i \leq k$ . If  $\lambda_1, \lambda_2, \dots, \lambda_k$  are different, then  $M$  is said to be of *k-type*.

The notion of finite type immersion is naturally extended to the Gauss map  $G$  on  $M$  by B.-Y. Chen and P. Piccinni [6]. Thus, a submanifold  $M$  of an Euclidean or pseudo-Euclidean space has *1-type Gauss map*  $G$ , if  $G$  satisfies  $\Delta G = a(G + C)$  for some  $a \in \mathbb{R}$  and some constant vector  $C$ .

However, the Laplacian of the Gauss map of some typical well-known surfaces in the three-dimensional Euclidean space  $\mathbb{E}^3$  such as the helicoid, the catenoid and the right cone takes a somewhat different form, namely,  $\Delta G = \phi(G + C)$  for some non-constant function  $\phi$

and some constant vector  $C$ . It looks like an eigenvalue problem, but the function  $\phi$  turns out to be non-constant. Therefore, it is worth studying the class of surfaces satisfying such an equation.

We use the following definition: a submanifold  $M$  of the Euclidean space  $\mathbb{E}^m$  or pseudo-Euclidean space  $\mathbb{E}_s^m$  is said to have *pointwise 1-type Gauss map* if its Gauss map  $G$  satisfies

$$\Delta G = \phi(G + C)$$

for some smooth function  $\phi$  on  $M$  and a constant vector  $C$ . A pointwise 1-type Gauss map is called *proper* if the function  $\phi$  is non-constant. A submanifold with pointwise 1-type Gauss map is said to be of *the first kind* if the vector  $C$  is zero. Otherwise, the pointwise 1-type Gauss map is said to be of *the second kind*.

In [10] M. Choi and Y. Kim characterized the helicoid in terms of pointwise 1-type Gauss map of the first kind. Together with B.-Y. Chen they proved that the class of surfaces of revolution with pointwise 1-type Gauss map of the first kind coincides with the class of surfaces of revolution with constant mean curvature and characterized the rational surfaces of revolution with pointwise 1-type Gauss map [4].

Tensor product surfaces with pointwise 1-type Gauss map and Vranceanu rotational surfaces with pointwise 1-type Gauss map in the four-dimensional Euclidean space  $\mathbb{E}^4$  were studied in [2] and [1], respectively.

In [21] Y. Kim and D. Yoon studied ruled surfaces with 1-type Gauss map in Minkowski space  $\mathbb{E}_1^m$  and gave a complete classification of null scrolls with 1-type Gauss map. The classification of ruled surfaces with pointwise 1-type Gauss map of the first kind in Minkowski space  $\mathbb{E}_1^3$  was given in [20]. Ruled surfaces with pointwise 1-type Gauss map of the second kind in Minkowski 3-space were classified in [11].

Recall that the Gauss map  $G$  of a submanifold  $M$  of  $\mathbb{E}^m$  is defined as follows. Let  $G(n, m)$  be the Grassmannian manifold consisting of all oriented  $n$ -planes through the origin of  $\mathbb{E}^m$  and  $\wedge^n \mathbb{E}^m$  be the vector space obtained by the exterior product of  $n$  vectors in  $\mathbb{E}^m$ .

In a natural way, we can identify  $\wedge^n \mathbb{E}^m$  with the Euclidean space  $\mathbb{E}^N$ , where  $N = \binom{m}{n}$ .

Let  $\{e_1, \dots, e_n, e_{n+1}, \dots, e_m\}$  be a local orthonormal frame field in  $\mathbb{E}^m$  such that  $e_1, e_2, \dots, e_n$  are tangent to  $M$  and  $e_{n+1}, e_{n+2}, \dots, e_m$  are normal to  $M$ . The map  $G : M \rightarrow G(n, m)$  defined by  $G(p) = (e_1 \wedge e_2 \wedge \dots \wedge e_n)(p)$  is called the *Gauss map* of  $M$ . It is a smooth map which carries a point  $p$  in  $M$  into the oriented  $n$ -plane in  $\mathbb{E}^m$  obtained by the parallel translation of the tangent space of  $M$  at  $p$  in  $\mathbb{E}^m$ .

In a similar way one can consider the Gauss map of a submanifold  $M$  of pseudo-Euclidean space  $\mathbb{E}_s^m$ .

For any function  $f$  on  $M$  the Laplacian of  $f$  is given by the formula

$$\Delta f = - \sum_i (\nabla'_{e_i} \nabla'_{e_i} f - \nabla'_{\nabla'_{e_i} e_i} f),$$

where  $\nabla'$  is the Levi-Civita connection of  $\mathbb{E}^m$  or  $\mathbb{E}_s^m$  and  $\nabla$  is the induced connection on  $M$ .

#### 4. MAIN RESULT

In this section we shall study marginally trapped surfaces with pointwise 1-type Gauss map.

Let  $M^2$  be a marginally trapped surface free of flat points and  $\{x, y, n_1, n_2\}$  be the geometric frame field of  $M^2$ , defined in Section 2. The geometric frame field  $\{x, y, n_1, n_2\}$  generates the following frame of the Grassmannian manifold:

$$\{x \wedge y, x \wedge n_1, x \wedge n_2, y \wedge n_1, y \wedge n_2, n_1 \wedge n_2\}.$$

The indefinite inner product on the Grassmannian manifold is given by

$$\langle e_{i_1} \wedge e_{i_2}, f_{j_1} \wedge f_{j_2} \rangle = \det (\langle e_{i_k}, f_{j_l} \rangle).$$

The Gauss map  $G$  of  $M^2$  is defined by  $G(p) = (x \wedge y)(p)$ ,  $p \in M^2$ . Then the Laplacian of the Gauss map is given by the formula

$$(3) \quad \Delta G = -\nabla'_x \nabla'_x G + \nabla'_{\nabla_x x} G - \nabla'_y \nabla'_y G + \nabla'_{\nabla_y y} G.$$

The derivative formulas (1) of  $M^2$  imply the following equalities for the invariants  $\nu, \lambda, \mu, \gamma_1, \gamma_2, \beta_1, \beta_2$  of the surface:

$$(4) \quad \begin{aligned} x(\mu) - 2\mu \gamma_2 - \mu \beta_1 &= 0; \\ y(\mu) - 2\mu \gamma_1 - \mu \beta_2 &= 0; \\ x(\lambda) - y(\nu) - 2\lambda \gamma_2 + 2\nu \gamma_1 + \lambda \beta_1 - (1 + \nu) \beta_2 &= 0; \\ x(\nu) + y(\lambda) - 2\lambda \gamma_1 - 2\nu \gamma_2 - (1 - \nu) \beta_1 + \lambda \beta_2 &= 0; \\ x(\beta_2) - y(\beta_1) + 2\nu \mu + \gamma_1 \beta_1 - \gamma_2 \beta_2 &= 0. \end{aligned}$$

Using equalities (1) and (3) we calculate the Laplacian of the Gauss map:

$$(5) \quad \begin{aligned} \Delta G = & -4\lambda \mu x \wedge y - (x(\lambda) - y(\nu) - 2\lambda \gamma_2 + 2\nu \gamma_1 + \lambda \beta_1 + (1 - \nu) \beta_2) x \wedge n_1 \\ & - (x(\mu) - 2\mu \gamma_2 - \mu \beta_1) x \wedge n_2 + (y(\mu) - 2\mu \gamma_1 - \mu \beta_2) y \wedge n_2 \\ & + (x(\nu) + y(\lambda) - 2\lambda \gamma_1 - 2\nu \gamma_2 + (1 + \nu) \beta_1 + \lambda \beta_2) y \wedge n_1 \\ & - 4\mu \nu n_1 \wedge n_2. \end{aligned}$$

Equalities (4) and (5) imply that the Laplacian of the Gauss map of a marginally trapped surface free of flat points is expressed in terms of the invariants of the surface by the following formula:

$$(6) \quad \Delta G = -4\lambda \mu x \wedge y - 2\beta_2 x \wedge n_1 + 2\beta_1 y \wedge n_1 - 4\mu \nu n_1 \wedge n_2.$$

Using (2) we can rewrite (6) in the form:

$$\Delta G = -2K x \wedge y - 2\beta_2 x \wedge n_1 + 2\beta_1 y \wedge n_1 + 2\kappa n_1 \wedge n_2,$$

where  $K$  and  $\kappa$  are the Gauss curvature and the normal curvature, respectively.

Now we shall find all marginally trapped surfaces with pointwise 1-type Gauss map.

**Theorem 4.1.** *Let  $M^2$  be a marginally trapped surface free of flat points. Then  $M^2$  is of pointwise 1-type Gauss map if and only if  $M^2$  has parallel mean curvature vector field.*

*Proof:* Let  $M^2$  be a marginally trapped surface free of flat points. Then the Laplacian of the Gauss map is expressed by formula (6).

In the case when the Gauss curvature is non-zero at a point  $p \in M^2$ , we have that  $\lambda \neq 0$  in a neighbourhood of  $p$ . Then the Laplacian of the Gauss map can be written as

$$(7) \quad \Delta G = -4\lambda \mu G - 4\lambda \mu T,$$

where  $T = \frac{\beta_2}{2\lambda\mu} x \wedge n_1 - \frac{\beta_1}{2\lambda\mu} y \wedge n_1 + \frac{\nu}{\lambda} n_1 \wedge n_2$ .

The condition that the surface has pointwise 1-type Gauss map is

$$\Delta G = \phi(G + C)$$

for some smooth function  $\phi$  on  $M^2$  and a constant vector  $C$ . Using (7) we obtain that  $M^2$  is of pointwise 1-type Gauss map if and only if  $T = \text{const}$ .

By the use of formulas (1) we calculate that

$$\begin{aligned} \nabla'_x T &= \frac{\beta_2}{2\lambda} x \wedge y + \left( x \left( \frac{\beta_2}{2\lambda\mu} \right) + \frac{\beta_1\beta_2 + \beta_1\gamma_1}{2\lambda\mu} - \frac{\nu(1+\nu)}{\lambda} \right) x \wedge n_1 \\ &\quad + \left( -x \left( \frac{\beta_1}{2\lambda\mu} \right) + \frac{\beta_2\gamma_1 - (\beta_1)^2}{2\lambda\mu} - \nu \right) y \wedge n_1 + \frac{\nu\mu}{\lambda} y \wedge n_2 \\ &\quad + \left( \frac{\beta_1}{2\lambda} + x \left( \frac{\nu}{\lambda} \right) \right) n_1 \wedge n_2; \\ (8) \quad \nabla'_y T &= \frac{\beta_1}{2\lambda} x \wedge y + \left( y \left( \frac{\beta_2}{2\lambda\mu} \right) + \frac{(\beta_2)^2 - \beta_1\gamma_2}{2\lambda\mu} - \nu \right) x \wedge n_1 \\ &\quad + \frac{\nu\mu}{\lambda} x \wedge n_2 + \left( -y \left( \frac{\beta_1}{2\lambda\mu} \right) - \frac{\beta_1\beta_2 + \beta_2\gamma_2}{2\lambda\mu} - \frac{\nu(1-\nu)}{\lambda} \right) y \wedge n_1 \\ &\quad + \left( y \left( \frac{\nu}{\lambda} \right) - \frac{\beta_2}{2\lambda} \right) n_1 \wedge n_2. \end{aligned}$$

Equalities (8) imply that  $T = \text{const}$  if and only if  $\beta_1 = 0$ ,  $\beta_2 = 0$ ,  $\nu = 0$  in this neighbourhood.

If  $M^2$  is flat, i.e.  $\lambda = 0$  at each point, then the Laplacian of the Gauss map is

$$\Delta G = -2T_0,$$

where  $T_0 = \beta_2 x \wedge n_1 - \beta_1 y \wedge n_1 + 2\mu\nu n_1 \wedge n_2$ . Using (1) we find

$$\begin{aligned} \nabla'_x T_0 &= \mu\beta_2 x \wedge y + (x(\beta_2) + \beta_1\beta_2 + \beta_1\gamma_1 - 2\mu\nu(1+\nu)) x \wedge n_1 \\ &\quad + (-x(\beta_1) + \beta_2\gamma_1 - (\beta_1)^2) y \wedge n_1 + 2\mu^2\nu y \wedge n_2 \\ &\quad + (x(2\mu\nu) + \mu\beta_1) n_1 \wedge n_2; \\ \nabla'_y T_0 &= \mu\beta_1 x \wedge y + (y(\beta_2) + (\beta_2)^2 - \beta_1\gamma_2) x \wedge n_1 \\ &\quad + 2\mu^2\nu x \wedge n_2 + (-y(\beta_1) - \beta_1\beta_2 - \beta_2\gamma_2 - 2\mu\nu(1-\nu)) y \wedge n_1 \\ &\quad + (y(2\mu\nu) - \mu\beta_2) n_1 \wedge n_2, \end{aligned}$$

which imply again that  $T_0 = \text{const}$  if and only if  $\beta_1 = 0$ ,  $\beta_2 = 0$ ,  $\nu = 0$ .

Marginally trapped surfaces satisfying  $\beta_1 = \beta_2 = 0$  have parallel mean curvature vector field, since  $DH = 0$  holds identically in view of (1). From (2) it follows that the equality  $\nu = 0$  is equivalent to  $\varkappa = 0$ , i.e. the surface is of flat normal connection. The last equality of (4) implies that all marginally trapped surfaces with parallel mean curvature vector field ( $\beta_1 = \beta_2 = 0$ ) have flat normal connection ( $\nu = 0$ ).

Finally we obtain that  $M^2$  is of pointwise 1-type Gauss map if and only if  $M^2$  has parallel mean curvature vector field.  $\square$

Note that the Laplacian of the Gauss map of each marginally trapped surface  $M^2$  with pointwise 1-type Gauss map is expressed as follows:

$$\Delta G = -4\lambda \mu G = -2K G,$$

where  $K$  is the Gauss curvature of  $M^2$ . Hence,  $M^2$  is of the first kind ( $C = 0$ ). Moreover,  $M^2$  is proper ( $\phi \neq \text{const}$ ) in the case of non-constant Gauss curvature.

The class of marginally trapped surfaces with parallel mean curvature vector field, was classified by B.-Y. Chen and J. Van Der Veken in [9]. Combining their classification with our result, we obtain all marginally trapped surfaces with pointwise 1-type Gauss map.

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